Density, short-range order and the quark-gluon plasma

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We study the thermal part of the energy density spatial correlator in the quark-gluon plasma. We describe its qualitative form at high temperatures. We then calculate it out to distances $\approx 1.5/T$ in SU(3) gauge theory lattice simulations for the range of temperatures $0.9 \leq T/T_c \leq 2.2$. The vacuum-subtracted correlator exhibits non-monotonic behavior, and is almost conformal by $2T_c$. Its broad maximum at $r \approx 0.6/T$ suggests a dense medium with only weak short-range order, similar to a non-relativistic fluid near the liquid-gas phase transition, where η/s is minimal.

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Hydrodynamics calculations [1] successfully described the pattern of produced particles in heavy ion collisions at RHIC [2]. This early agreement between ideal hydrodynamics and experiment has been refined in recent times. On the theory side, the dissipative effects of shear viscosity η have been included in full 3d hydrodynamics calculations [3, 4, 5] and the sensitivity to initial conditions quantitatively estimated [6] for the first time. On the experimental side, the elliptic flow observable v_2, which is sensitive to the value of η in units of entropy density s, is now corrected for non-medium-generated two-particle correlations [7]. The conclusion that η/s must be much smaller than unity has so far withstood these refinements of heavy-ion phenomenology [6].

The smallness of η/s was turned into the statement that the quark-gluon plasma (QGP) formed at RHIC is the "most perfect liquid known in nature" [8]. A general question then comes to mind: what observable can be used to characterize the liquid nature of a system described by a quantum field theory [9]? And secondly, what is the QCD prediction for that observable? This leads us to remind ourselves what the defining property of an ordinary liquid is. Surely the everyday-life notion that a liquid "has a definite volume, but no definite shape" is inadequate in the present context.

The two-body density distribution $\rho(\mathbf{r_1}, \mathbf{r_2}) = g(r)\rho^2$ of an ordinary substance (such as water) of density ρ behaves qualitatively differently in the solid, liquid and gas phase (see for instance [10]). The radial distribution function $g(r)$ characterizes the average density of particles at distance r from an arbitrarilty chosen particle. In a dilute gas, $g(r)$ is essentially equal to 1 for r greater than the size of a molecule. In a liquid on the other hand, $g(r)$ vanishes at small r, a reflexion of the short-distance repulsion between molecules. The function then rises and typically exhibits several gradually damped oscillations around unity. This reflects the "short-range order" in the fluid, namely the coherent motion of closely packed molecules up to distances a few times the molecule size. Over longer distances, this ordering is lost. Only a perfect crystal at low temperatures exihibits truly long-range order.

In quantum field theory, particle number is not (necessarily) conserved, so it is not immediately clear which spatial correlator is the closest analogue of the two-body density distribution in non-relativistic systems. In QCD, the only conserved quantities are energy, momentum, the quark numbers and the (non-singlet) axial charges. The energy density is an order parameter in the theoretical limit of a large number of colors N_c, which puts it in natural correspondence with the density ρ of a non-relativistic fluid. Hence our strategy to study the spatial correlator of the energy density.

A peculiarity of quantum field theory is that energy density correlations are present even when the average energy density is zero, i.e. at $T = 0$ when the partition function is saturated by the quantum vacuum. This correlation has to be strong at short distance in an asymptotically free theory, $\langle T_{00}(0)T_{00}(\mathbf{r})\rangle \sim r^{-8}$, on dimensional grounds. Based on the Källen-Lehmann representation, it is monotonically decreasing in r at any T.

Therefore, in order to isolate the thermal effects, we shall consider the subtracted correlator

$$G_{ee}(T, r) = \langle T_{00}(0)T_{00}(0, \mathbf{r})\rangle_T - \langle T_{00}(0)T_{00}(0, \mathbf{r})\rangle_{T=0}, \tag{1}$$

In the Euclidean SU(N_c) gauge theory, the energy density operator,

$$\begin{aligned} T_{00} &= \theta_{00} + \frac{1}{4}\theta\,, \quad \theta = \frac{\beta(g)}{2g} F^a_{\mu\nu}F^a_{\mu\nu}, \quad \langle\theta\rangle = e - 3p, \quad (2) \\ \theta_{00} &= \frac{1}{4}F^a_{ij}F^a_{ij} - \frac{1}{2}F^a_{0i}F^a_{0i}\,, \quad \langle\theta_{00}\rangle = \frac{3}{4}(e+p) = \frac{3}{4}T\,s\,, \end{aligned}$$

is composed of two terms which are separately scale-independent operators. Here e is the energy density, p the pressure and $\beta(g) = -bg^3 + \ldots$ the beta function, with $b = \frac{11N_c}{3(4\pi)^2}$. The expectation value of the 'naive' energy operator θ_{00} is proportional to the entropy density, while the expectation value of the trace anomaly directly measures the deviation from the conformal limit

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where $e = 3p$ (we implicitly add a constant to T_{00} such that $\langle T_{00}\rangle$ vanishes in the vacuum). We can thus investigate separately the G_{ss} and $G_{\theta\theta}$ correlators, defined by replacing T_{00} respectively by $\frac{4}{3}\theta_{00}$ and θ in Eq. 1.

High-temperature behavior

The one-loop expression for the two correlators in D space-time dimensions is

$$\langle\theta_{00}(0)\theta_{00}(x)\rangle_{1\mathrm{L}} = (8\pi b\alpha_s)^{-2}\langle\theta(0)\theta(x)\rangle_{1\mathrm{L}} = \qquad (3)$$
$$\frac{d_A}{4\pi^4}\sum_{m,n\in\mathbf{Z}}\left(D-8+16\frac{(x_{[m]}\cdot x_{[n]})^2}{x_{[m]}^2 x_{[n]}^2}\right)\frac{1}{(x_{[m]}^2\, x_{[n]}^2)^2}$$

where $d_A = N_c^2 - 1$ and $x_{[n]} = (\frac{n}{T} + x_0, \mathbf{x})$, while $\langle\theta_{00}(0)\theta(0,\mathbf{r})\rangle_{1\mathrm{L}}$ vanishes identically. The $m = n = 0$ term gives the zero-temperature expression.

Let us consider the high-temperature regime, $T \gg g^2(T)T \gg T_c$. In that case, the behavior of G_{ee} is characterized by three regimes:

$Tr \ll g^{-2}(T)$: both the zero and high temperature correlators are well described by the one-loop formula, up to small radiative corrections. For $r \to 0$, $G_{ee}(T,r) \sim -\frac{2T^4}{45r^4}$; for $Tr > 0.568$, $G_{ee} > e^2(T)$ (see Fig. 3).

$g^{-2}(T) \ll Tr \ll \frac{T}{T_c}$: the zero temperature correlator is still described by perturbation theory, $\frac{3d_A}{\pi^4 r^8}$, but the high-T correlator is exponentially screened, so necessarily $G_{ee} < e^2(T)$.

$Tr \gg \frac{T}{T_c}$: both the zero and high temperature correlators are exponentially screened. In the former case, the relevant mass is $M_4 \simeq 5.3T_c$ [11, 12], corresponding to the lightest scalar glueball mass in $D = 4$, while at high temperatures, dimensional reduction takes places, and the screening mass is $(M_3/g_3^2)\cdot g^2(T)T$ with $M_3/g_3^2 \simeq 2.4$ [13]. It is therefore clear that screening of the energy density is stronger at high temperatures, and hence G_{ee} asymptotically approaches $e^2(T)$ from below at a rate $e^{-M_4 r}$.

In summary, at high temperatures $G_{ee}(T,r) - e^2(T)$ vanishes at least at two finite distances r: the first time at $Tr \approx 0.568$, and the second time at $Tr = \mathrm{O}(g^{-2}(T))$. To investigate the function $G_{ee}(T,r)$ at temperatures accessible in heavy-ion colliders, we perform lattice simulations in the region $0.9T_c < T < 2T_c$.

Operator product expansion

From expression (3) and from known results [14], we can obtain the leading terms in the operator-product expansion (OPE) of these correlators,

$$\langle\theta(0)\theta(x)\rangle \sim \frac{(8\pi b\alpha_s)^2 3d_A}{\pi^4 r^8} - \frac{64b^2}{3}\alpha_s^2\frac{\langle\theta_{00}\rangle}{r^4} - 32b^2\alpha_s^2\frac{\langle\theta\rangle}{r^4}$$
$$\langle\theta_{00}(0)\theta_{00}(x)\rangle \sim \frac{3d_A}{\pi^4 r^8} - \frac{1}{3\pi^2}\frac{\langle\theta_{00}\rangle}{r^4} + \mathrm{O}(\alpha_s^0)\frac{\langle\theta\rangle}{r^4}\,, \qquad (4)$$

where $x = (0,\mathbf{r})$ and r^{-2} terms and softer have been omitted. The Wilson coefficients of the operators $\mathbf{1}$, θ_{00} and θ are a least of $\mathrm{O}(\alpha_s^2)$ for the product $\theta_{00}(0)\theta(0,\mathbf{r})$. The usefulness of the OPE in this context arises because of the exact cancellation of the r^{-8} term in the difference between finite T and $T=0$ correlators. Using Eq. 4, we obtain the short-distance behavior

$$G(T,\mathbf{r}) \sim -\frac{e+p}{(2\pi r^2)^2} + \mathrm{O}(\alpha_s^0)\frac{e-3p}{r^4} + \mathrm{O}(\alpha_s r^{-4}, r^{-2}) \quad (5)$$

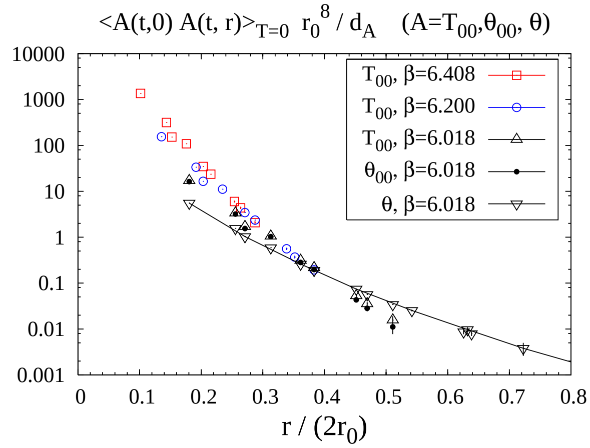

FIG. 1: The correlators at $T = 0$ at different lattice spacings ($r_0 \approx 0.5$fm [18]; the line is to guide the eye).

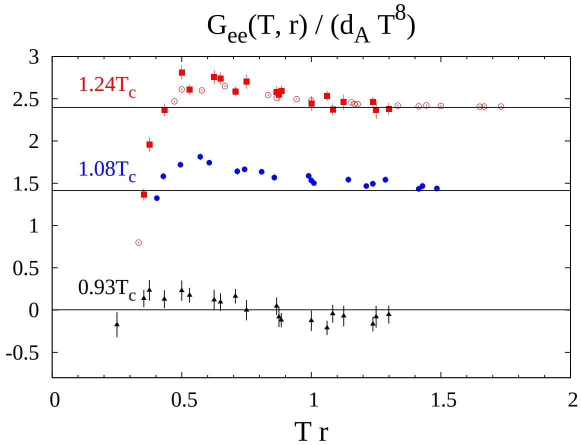

FIG. 2: G_{ee} (Eq. 1) across the deconfining phase transition, at $\beta = 6.018$. At $T = 1.24T_c$ we check for discretization errors by also showing data from $\beta = 6.200$ (filled squares).

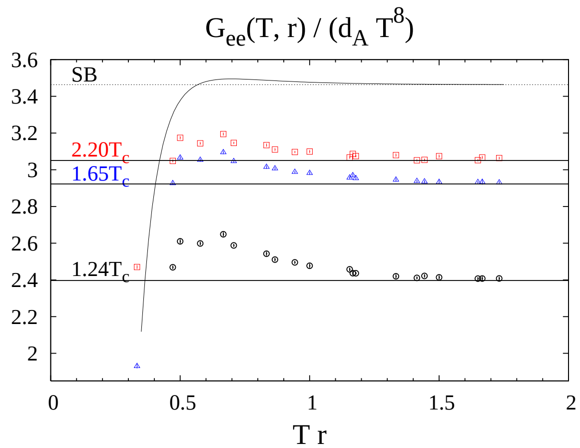

FIG. 3: G_{ee} at three temperatures, with $N_\tau = 6$. The curve corresponds to the correlator in the Stefan-Boltzmann limit $T \to \infty$.

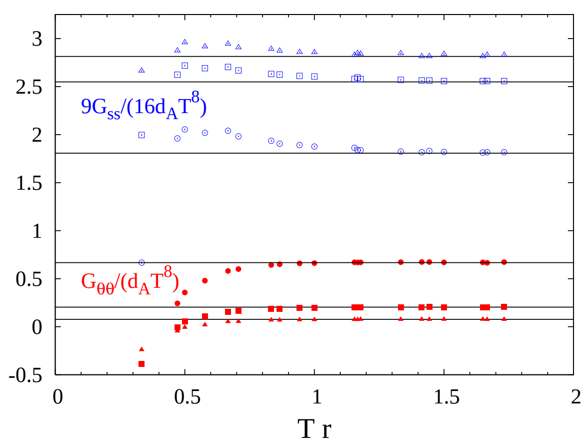

FIG. 4: The $N_\tau = 6$ entropy-entropy (G_{ss}) and action-action ($G_{\theta\theta}$) correlators at 1.24 (circles), 1.65 (squares) and $2.20T_c$ (triangles).

Numerical results

We now turn to a calculation of $G_{ee}(T, r)$ in the SU(3) gauge theory, i.e. in the plasma of gluons, using lattice Monte-Carlo techniques on a $(1/T) \cdot L^3$ lattice. We employ the (isotropic) Wilson action [15] and the 'once-HYP-smeared' 'clover' discretization of θ_{00} and θ developed in [16]. The variance of $\langle \theta_{00} \rangle$ was shown [17] to be reduced by almost two orders of magnitude as compared to the simplest 'plaquette' discretization. This technical improvement allows us to obtain a signal for $G_{ee} - e^2(T)$ out to $r \simeq 1.5/T$.

Figure 1 displays the relevant correlators at zero temperature. They fall off monotonically as r^{-8} at short distance and exponentially at large distance. Note that the trace anomaly correlator, while $O(\alpha_s^2)$ at short distances, dominates at large distances. For $r/a \geq 3$ comparison of the data obtained at three lattice spacings shows that discretization errors are under control.

Figure 2 shows the qualitative change of G_{ee} across the deconfining phase transition. At $T = 1.24T_c$ the functions obtained from these two lattice spacings are in qualitative agreement, a non-trivial check, given the large cancellation between the finite and zero temperature correlators. The function $G_{ee}(r)$ is large and negative at short-distances, crosses the asymptotic value $e^2(T)$, reaches a maximum and presumably decreases monotonically after that. Although the signal becomes too small to tell beyond $Tr \simeq 1.5$, this is plausible in view of the small value of the thermal screening mass (see next section).

The $1/r^4$ short-distance divergence is not visible below T_c, a fact that the OPE and the smallness of (e, p) in the confined phase easily account for. We note that $G_{ee}/e^2(T)$ reaches around $r = 0.6/T$ a maximum which is larger at 1.08 than at $1.24T_c$. We understand this in terms of the larger fluctuations present near the (weakly) first order phase transition.

Figure 3 shows the temperature dependence of G_{ee} up to $2.2T_c$. The position of the maximum remains $r_{\max} \approx 0.6/T$, and $G_{ee}(r_{\max}, T)/e^2(T)$ decreases slowly as the temperature rises. The curves at 1.65 and $2.20T_c$ exhibit near-conformal behavior (i.e., G_{ee} is essentially a function of Tr), however the asymptotic approach to $e^2(T)$ has the opposite sign, as a study of screening masses shows.

Figure 4 shows separately the entropy density correlator and the trace anomaly correlator. The former is qualitatively similar to the energy density correlator, while the latter has a rather featureless monotonic behavior.

Screening masses

The asymptotic large-r behavior of G_{ee} is dictated by the smallest screening mass that T_{00} couples to. This is the state invariant under all the symmetries of a constant 'z-slice' [19, 20], which has a volume $(1/T) \times L \times L$. From $D = 4$ simulations, we obtain directly

$$\frac{M(T)}{M_4} = 0.630(14), \quad 0.906(20), \quad 1.276(32) \qquad (6)$$

respectively at 1.24 ($N_\tau = 8$), 1.65, and $2.20T_c$ ($N_\tau = 6$). So it is only at $T^* = 1.790(36)T_c$ that the thermal screening starts to exceed the $D = 4$ glueball mass. In particular, for $T > T^*$, $G_{ee}(r, T)$ approaches its asymptotic value from below, and therefore crosses $e^2(T)$ twice.

Comparison with non-relativistic systems

The radial distribution function $g(r)$ of a simple non-relativistic liquid, such as ^{36}Ar at 85K [21], exhibits several very pronounced peaks above 1. In particular $g(r) - 1$ is of order unity at the first peak. However, it is

known [22] that η/s is minimal near the liquid-gas phase transition, and becomes large both at low and high temperature. A highly ordered mesoscopic scale favours the transport of momentum, because the holes between the closely packed molecules then play the role of quasiparticles with a long mean free path (an argument attributed to Enskog [22]). Heating up the liquid has the effect of reducing the amplitude of the peaks in $g(r)$, until they disappear completely once the system is in a dilute gas phase. Thus the regime where η/s is minimal is the one where $g(r)$ has only few, small oscillations around unity.

We have found that G_{ee} has exactly one broad peak above $e^2(T)$, exceeding that value by about $5-15\%$ (the figure decreases slowly with temperature). By analogy with non-relativistic fluids, it is tempting to see a relation between this fact and the small value obtained for the shear viscosity [23] in the same range of temperatures, $\eta/s < 1.0$.

Conclusion

We have calculated non-perturbatively the thermal part G_{ee} of the energy-density spatial correlator in the plasma of SU(3) gluons, in the range of temperatures $0.9 \le T/T_c \le 2.2$, and found the qualitative high-T behavior. It diverges as $-r^{-4}$ near the origin as dictated by the OPE, and above T_c reaches a maximum at $r_{\max} \approx 0.6/T$ typically 10% above its asymptotic value $e^2(T)$. For $T < 1.8T_c$ it asymptotically approaches that value from above, and at higher temperatures it approaches from below. While the appearance of G_{ee} is rather different from the radial distribution function of a typical liquid, we pointed out that it is precisely when the short-range order is weak that the ratio η/s is minimal.

We note that the radial distribution function of monopoles has been computed in SU(2) gauge theory [24], and has a similar shape to G_{ee}. Color charge and monopole-antimonopole correlators that look alike have also been found in models [25]. It would be interesting to see whether such models can reproduce G_{ee}.

A straightforward extension of this work is to consider the full spatial correlations of the energy momentum tensor $\langle T_{\mu\nu}T_{\rho\sigma}\rangle$, in SU(3) gauge theory and in full QCD. As a benchmark it would be helpful to know the form of these correlators in the strongly coupled $\mathcal{N} = 4$ SYM theory, which is known to be an excellent fluid from the smallness of η/s [26] and from its spectral functions [27].

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